Short Papers

TE and TM Modes of Some Triangular Cross-Section Waveguides Using Superposition of Plane Waves

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Abstract —Exact transverse electric and magnetic mode solutions of four triangular cross-section waveguides have been found via a new general method using Snell's law and superposition of plane waves. This paper presents results for 1) equilateral, 2) 30°, 30°, 120°, 3) isosceles right, and 4) 30°, 60° right triangular waveguides. The electric and magnetic field solutions form finite sums of separable rectangular harmonics and are the only waveguides of triangular cross section for which such solutions have been found.

I. Introduction

It is possible to superpose four plane waves of the form

$$\vec{E}_i = \vec{E}_{0_i} \exp(-jk_0 \hat{k}_i \cdot \vec{r}), \qquad i = 1,2,3,4 \tag{1}$$

and find the complete set of transverse electric (TE) and transverse magnetic (TM) modes of a rectangular waveguide with perfectly conducting walls [1]. These plane waves are reflections of one initial wave from each of the waveguide walls. They are related by Snell's law and possess equal amplitudes but different phases. Thus, beginning with an arbitrary plane wave, it is possible to generate a complete set (which may be infinite) of wave propagation vectors which characterizes many waveguides of polygonal cross section and derive exact expressions for the longitudinal components of the electric and magnetic fields by superimposing waves in the form of (1).

These fields must obey the required boundary conditions for perfect conductors, i.e., zero tangential electric field or zero normal derivative of the longitudinal magnetic field on the boundaries. If these boundary conditions can be satisfied, then the superimposed plane waves provide some of the possible waveguide modes.

The limitation of attempting the solution of an N-sided cross-section waveguide resides in the tedious algebra associated with computing the wave propagation vectors and applying the boundary conditions for the electric and magnetic fields. Computer algebra [2] has been helpful in this regard, but we have restricted our present investigations to those specific triangular cross-section waveguides which give the simplest types of solutions [3].

Using a new general method based solely on Snell's law and superposition of plane waves, complete sets of TE and TM modes for triangular waveguides have been determined for the following cross sections: 1) equilateral, 2) 30°, 30°, 120°, 3) isosceles right, and 4) 30°, 60° right. Solutions of these four cross sections form finite sums of rectangular harmonics and are the only triangular waveguide solutions which have been found.

Analytic solutions for the isosceles right [4], [5] and equilateral [6] triangular waveguides have been developed previously using specialized techniques which are unique to a particular geometry. The solution for the isosceles right cross section provides a complete mode set, but the equilateral solution given by Schelkunoff does not provide a complete mode set, i.e., the "odd" modes are missing. This point will be elaborated in Section III and the Appendix.

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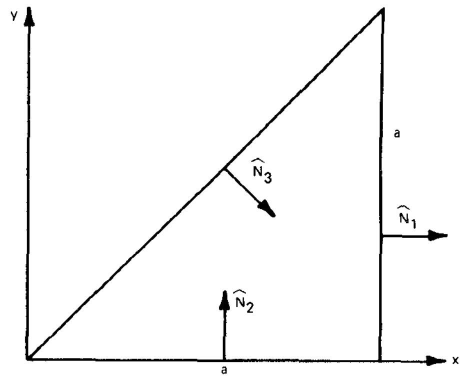

Fig. 1. Cross section of isosceles right triangular waveguide.

This method can be extended and used for a number of polygonal waveguides and can provide closed analytic expressions for the possible modes, although complete sets are not always obtained. These solutions provide a check on the accuracy of eigenfunctions and cutoff frequencies obtained when approximate numerical techniques [7]–[15] are applied to more general wavguide cross sections.

In Section II, the wave vectors are presented, exact expressions for the TE and TM modes with associated eigenvalues are determined, and the lowest order cutoff wavenumbers are computed for each of the four special cross sections.

In Section III, some three-dimensional and contour plots of dominant modes are presented and discussed. Also in Section III and the Appendix, the equilateral triangular waveguide solutions are compared with those of Schelkunoff showing that they (both TE and TM) do not provide a complete mode set.

II. Theory

We introduce an initial wave vector of the form

$$\hat{k}_i = \alpha\hat{X} + \beta\hat{Y} + \gamma\hat{Z} = \frac{\vec{k}_i}{|\vec{k}_i|} \tag{2}$$

where α, β, and γ are the direction cosines of the vector. Using Snell's law in vector form [16], the wave vector $\hat{k}_r$ reflected from the jth wall is

$$\hat{k}_r = \hat{k}_i - 2\hat{N}_j(\hat{k}_i \cdot \hat{N}_j) \tag{3}$$

where $\hat{N}_j$ is the unit normal to the jth wall and $\hat{k}_i$ is the incident unit wave vector given by (2).

Using this incident wave vector, the set of reflected unit wave vectors can be generated by iterating (3). For the special geometries considered, these sets are finite. For arbitrary interior

TABLE I
TM AND TE MODES FOR FOUR TRIANGULAR WAVEGUIDES

Triangular cross section	TM modes	TE modes	Eigenvalues (m, n integer)
Isosceles right	$E_z = \sin k_1x \sin k_2y$	$H_z = \cos k_1x \cos k_2y$	$k_1 = m\pi/a;\ k_2 = n\pi/a$
	$- \sin k_2x \sin k_1y$	$+ \cos k_2x \cos k_1y$	
30°-60° right	$E_z = \sin k_1x \sin k_2y$	$H_z = \cos k_1x \cos k_2y$	$k_1 = m\pi/a;\ k_2 = n\pi/a\sqrt{3}$
	$+ \sin k_3x \sin k_4y$	$+ \cos k_3x \cos k_4y$	$k_3 = (m-n)\pi/2a;\ k_4 = (n+3m)\pi/2a\sqrt{3}$
	$+ \sin k_5x \sin k_6y$	$+ \cos k_5x \cos k_6y$	$k_5 = (m+n)\pi/2a;\ k_6 = (n-3m)\pi/2a\sqrt{3}$
Equilateral	$E_z^{(1)} = \sin 2k_1x \sin 2k_2y$	$H_z^{(1)} = \cos 2k_1x \cos 2k_2y$	$k_1 = 2m\pi/a\sqrt{3};\ k_2 = 2n\pi/3a$
	$+ \sin k_3x \sin k_4y$	$+ \cos k_3x \cos k_4y$	$k_3 = (m+n)2\pi/a\sqrt{3};\ k_4 = (n-3m)2\pi/3a$
	$+ \sin k_5x \sin k_6y$	$+ \cos k_5x \cos k_6y$	$k_5 = (m-n)2\pi/a\sqrt{3};\ k_6 = (n+3m)2\pi/3a$
	$E_z^{(2)} = \sin 2k_1x \cos 2k_2y$	$H_z^{(2)} = \cos 2k_1x \sin 2k_2y$	
	$- \sin k_3x \cos k_4y$	$- \cos k_3x \sin k_4y$	
	$- \sin k_5x \cos k_6y$	$- \cos k_5x \sin k_6y$	
Isosceles (120°)	$E_z = \sin k_1x \sin k_2y$	$H_z = \cos k_1x \cos k_2y$	$k_1 = 2m\pi/a;\ k_2 = 2n\pi/a\sqrt{3}$
	$- \sin k_3x \sin k_4y$	$+ \cos k_3x \cos k_4y$	$k_3 = (m+n)\pi/a;\ k_4 = (3m-n)\pi/a\sqrt{3}$
	$+ \sin k_5x \sin k_6y$	$+ \cos k_5x \cos k_6y$	$k_5 = (m-n)\pi/a;\ k_6 = (3m+n)\pi/a\sqrt{3}$

Isosceles right: TM modes; $m \neq 0,\ n \neq 0,\ m \neq n$

$$TM_{mn} = TM_{nm}$$

TE modes; $TE_{mn} = TE_{nm}$

30, 60° right: Coordinate system as in Fig. 1 with 30° angle at origin.
TM modes; $m \neq 0,\ n \neq 0,\ m \neq n,\ n \neq 3m$

$$TM_{31} = TM_{24} = TM_{15}$$

TE modes; $TE_{02} = TE_{11},\ TE_{13} = TE_{20}$
All modes; $(m+n)$ and $(m-n)$ are even

Equilateral: $TM^{(1)}$ modes; $m \neq 0,\ n \neq 0,\ m \neq n,\ n \neq 3m$
$TM^{(2)}$ modes; $m \neq 0,\ m \neq n$
$TE^{(1)}$ modes; ---- $TE_{11}^{(1)} = TE_{02}^{(1)}$
$TE^{(2)}$ modes; $n \neq 0$ ($TE_{10}^{(2)}$ does not exist), $n \neq 3m$

120° isosceles: Coordinate system as in Fig. 2 with 120° angle at origin and adjacent (equal) sides of length a.
TM modes; $m \neq 0,\ n \neq 0,\ m \neq n,\ n \neq 3m$

$$TM_{31} = TM_{15} = TM_{24}$$

TE modes; $TE_{02} = TE_{11},\ TE_{13} = TE_{20},\ TE_{22} = TE_{04}$
All modes; $(m+n)$ and $(m-n)$ are even.

angles, the resulting sets are unbounded. Each vector will have the same z-component γ, since the wall normals $\hat{N}_j$ have only x and y components. The direction cosine γ determines the phase velocity of the wave down the guide and is not shown explicitly in what follows.

Having determined the complete set of propagation vectors, the z-component of the electric field can be written as

$$\vec{E}_z = \sum_{i=1}^{N} \vec{E}_{oi} \exp(-jk_o\hat{k}_i \cdot \vec{r}) \tag{4}$$

where N is the total number of wave vectors, $\vec{r} = x\hat{X} + y\hat{Y}$, and where $E_z = 0$ on the waveguide boundary. The z-component of the magnetic field can be written similarly, subject to $\partial H_z/\partial n = 0$ on the boundary.

For example, the TE and TM modes for an isosceles right triangular waveguide have been found as follows (see Fig. 1). The wall normals are

$$\hat{N}_1 = \hat{X}, \quad \hat{N}_2 = \hat{Y}, \quad \hat{N}_3 = (\hat{X} - \hat{Y})/\sqrt{2}. \tag{5}$$

Substituting (2) and (5) into (3), the complete set of transverse wave propagation vectors is determined

$$\begin{aligned} \hat{k}_1 &= \alpha\hat{X} + \beta\hat{Y} & \hat{k}_5 &= \beta\hat{X} + \alpha\hat{Y} \\ \hat{k}_2 &= -\alpha\hat{X} + \beta\hat{Y} & \hat{k}_6 &= -\beta\hat{X} + \alpha\hat{Y} \\ \hat{k}_3 &= \alpha\hat{X} - \beta\hat{Y} & \hat{k}_7 &= \beta\hat{X} - \alpha\hat{Y} \\ \hat{k}_4 &= -\alpha\hat{X} - \beta\hat{Y} & \hat{k}_8 &= -\beta\hat{X} - \alpha\hat{Y}. \end{aligned} \tag{6}$$

TABLE II
SUMMARY OF WAVE PROPAGATION VECTORS FOR FOUR TRIANGULAR WAVEGUIDES

Triangular cross section	Number of $\hat{k}$ vectors	Form of $\hat{k}$ vectors[a]	Rectangular harmonic sum
Isosceles right	8	$[\alpha\hat{X},\beta\hat{Y}]$; $[\beta\hat{X},\alpha\hat{Y}]$	2
30–60° right	12	$[\alpha\hat{X},\beta\hat{Y}]$; $1/2[(\alpha-\beta\sqrt{3})\hat{X},\ (\beta+\alpha\sqrt{3})\hat{Y}]$; $1/2[(\alpha+\beta\sqrt{3})\hat{X},\ (\beta-\alpha\sqrt{3})\hat{Y}]$	3
Equilateral	12(6)[b]	$[\alpha\hat{X},\beta\hat{Y}]$; $1/2[(\alpha-\beta\sqrt{3})\hat{X},\ (\beta+\alpha\sqrt{3})\hat{Y}]$; $1/2[(\alpha+\beta\sqrt{3})\hat{X},\ (\beta-\alpha\sqrt{3})\hat{Y}]$	3
Isosceles	12	$[\alpha\hat{X},\beta\hat{Y}]$; $1/2[(\alpha-\beta\sqrt{3})\hat{X},\ (\beta+\alpha\sqrt{3})\hat{Y}]$; $1/2[(\alpha+\beta\sqrt{3})\hat{X},\ (\beta-\alpha\sqrt{3})\hat{Y}]$	3

[a]Notation means all the sign permutations of the coefficients of $\hat{X}$ and $\hat{Y}$ to form a vector. Hence, $[\alpha\hat{X},\beta\hat{Y}]$ means all of the vectors in the left-hand column of (6), while $[\beta\hat{X},\alpha\hat{Y}]$ gives the right-hand set of vectors.
[b]See text.

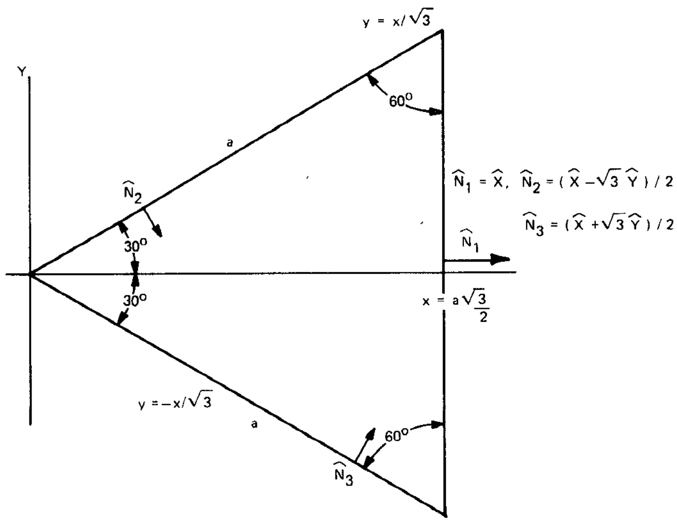

Fig. 2. Cross section of equilateral triangular waveguide.

Substitution into (4) gives E_z for the TM modes

$$\begin{aligned}E_z = {} & E_{01}e^{-jk_0\alpha x}e^{-jk_0\beta y} + E_{02}e^{jk_0\alpha x}e^{-jk_0\beta y}\\ & + E_{03}e^{-jk_0\alpha x}e^{jk_0\beta y} + E_{04}e^{jk_0\alpha x}e^{jk_0\beta y}\\ & + E_{05}e^{-jk_0\beta x}e^{-jk_0\alpha y} + E_{06}e^{jk_0\beta x}e^{-jk_0\alpha y}\\ & + E_{07}e^{-jk_0\beta x}e^{jk_0\alpha y} + E_{08}e^{jk_0\beta x}e^{jk_0\alpha y}\end{aligned} \tag{7}$$

where the boundary conditions are $E_z = 0$ on $y = 0$, $x = a$, $x = y$ (see Fig. 1). The first condition requires $E_{03} = -E_{01}$, $E_{04} = -E_{02}$, $E_{07} = -E_{05}$, $E_{08} = -E_{06}$ for all x. The second condition is satisfied by $E_{02} = -E_{01}\exp(-2jk_0\alpha a)$ and $E_{06} = -E_{05}\exp(-2jk_0\beta a)$, while the last condition requires $E_{01} = E_{05}$ and

$$k_0\alpha = m\pi/a = k_1, \quad k_0\beta = n\pi/a = k_2 \tag{8}$$

where m and n are integers. Substitution of these results into (7) gives

$$E_z = \sin k_1 x \sin k_2 y - \sin k_2 x \sin k_1 y \tag{9}$$

which determines the TM modes for the isosceles right triangular waveguide. A similar process for the TE modes yields

$$H_z = \cos k_1 x \cos k_2 y + \cos k_2 x \cos k_1 y \tag{10}$$

where the appropriate boundary conditions are $\partial H_z/\partial y = 0$ on $y = 0$, $\partial H_z/\partial x = 0$ on $x = a$, and $\partial H_z/\partial x - \partial H_z/\partial y = 0$ on $x = y$. Applying the method of superposition of plane waves outlined above, closed-form mode solutions of other triangular cross sections have been expressed as sums of rectangular harmonics. These solutions are given in Table I. The corresponding wave propagation vectors are given in Table II.

TABLE III
CUTOFF WAVENUMBER FOR FOUR TRIANGULAR WAVEGUIDES

Cross section	Modes (mn)	$(k_c)_{mn}/(k_c)^a_{\ell o}$	$(k_c)^a_{\ell o}$
Isosceles right	TE_{01}	1	π/a
	TE_{11}	1.414	
	TE_{02}	2	
	TE_{12}, TM_{12}	2.236	
30-60° Right	TE_{11}, TE_{02}[b]	1	$2\pi/a\sqrt{3}$
	TE_{13}, TE_{20}	1.732	
	TE_{22}, TE_{04}	2	
	TE_{31}, TM_{31}, TE_{24}, TM_{24}, TE_{15}, TM_{15}	2.646	
Equilateral	$TE^{(1)}_{01}$, $TE^{(2)}_{01}$[c]	1	$4\pi/3a$
	$TE^{(1)}_{10}$, $TM^{(2)}_{10}$	1.732	
	$TE^{(1)}_{11}$, $TE^{(2)}_{11}$, $TE^{(1)}_{02}$, $TE^{(2)}_{02}$	2	
	$TE^{(1)}_{12}$, $TM^{(1)}_{12}$, $TE^{(2)}_{12}$, $TM^{(2)}_{12}$	2.646	
30-30-120°	TE_{11}, TE_{02}	1	$4\pi/a\sqrt{3}$
	TE_{13}, TE_{20}	1.732	
	TE_{22}, TE_{04}	2	
	TE_{31}, TM_{31}, TE_{24}, TM_{24}, TE_{15}, TM_{15}	2.646	

[a] ℓo stands for lowest order mode
[b] Modes which have the same superscript (or no superscript) and the same cutoff wavenumber are actually the same mode regardless of their m, n indices.
[c] Modes which have different superscripts, the same m, n indices (either TE or TM), and the same cutoff wavenumbers are degenerate modes as in the case of the square waveguide.

The general unit propagation vector for any right triangular cross section is

$$\hat{k}_n = (\pm\alpha\cos 2n\phi \pm \beta\sin 2n\phi)\hat{X} + (\pm\alpha\sin 2n\phi \pm \beta\cos 2n\phi)\hat{Y}, \qquad n = 0,1,2,\cdots \tag{11}$$

where ϕ is one of the interior angles. The eight possible sign combinations represented by (11) can be grouped as two subsets of four, in each of which three signs are the same. The resulting sets are

$$\{(+++-)\ \ (++-+)\ \ (--+-)\ \ (---+)\}$$

and

$$\{(+-++)\ \ (+---)\ \ (-+++)\ \ (-+--)\}.$$

Each subset, when reduced, becomes a sum of rectangular harmonic terms, e.g., $\sin k_1x \sin k_2y$. For an arbitrary interior angle ϕ, each wave vector $\hat{k}_n$ is distinguishable and the general solution would require an infinite set. Two wave vectors, i.e., $\hat{k}_n$ and $\hat{k}_{n'}$ $(n \neq n')$, are indistinguishable when the sine/cosine arguments in (11) differ by a multiple of 2π. This degeneracy occurs for those values of the interior angle given by $\phi = \pi/m$, where m is an integer. For these discrete angles, the solutions can be represented by a finite set of rectangular harmonics.

Using the initial wave vector (2), the two subsets of four wave vectors which result for the right triangle become a single set of six for the equilateral. Using the initial wave vector, $\hat{k}_t = \alpha\hat{x} - \beta\hat{y} + \gamma\hat{z}$ generates a second set of six vectors which is given in Table II. Imposing the boundary conditions for TM modes, the resulting electric field expressions obtained from each wave vector set, E_{z_1} and E_{z_2}, respectively, are given by

$$E_{z_1} = E_{01}\left(e^{j2k_2y}\sin 2k_1x - e^{-jk_4y}\sin k_3x - e^{-jk_6y}\sin k_5x\right) \tag{12a}$$

and

$$E_{z_2} = E_{02}\left(e^{-j2k_2y}\sin 2k_1x - e^{jk_4y}\sin k_3x - e^{jk_6y}\sin k_5x\right) \tag{12b}$$

with respect to the coordinate system shown in Fig. 2. Real solutions are obtained by taking the linear combination $E_z = E_{z_1} \pm E_{z_2}$ and are given in Table I. The + and − solutions correspond to modes of even and odd symmetry, but have the same phase velocity. Similar results are obtained for the TE modes.

The cutoff wavenumbers are found by setting $\gamma = 0$ in (2), giving

$$\alpha^2 + \beta^2 = 1. \tag{13}$$

This relationship for the isosceles right triangular waveguide, together with (8), gives

$$(k_c)^2_{mn} = \pi^2(m^2 + n^2)/a^2 \tag{14}$$

for the cutoff wavenumber. Cutoff wavenumbers for the other

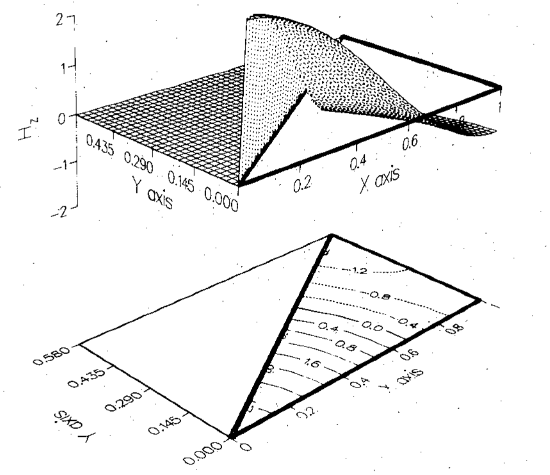

Fig. 3. TE_{11} mode for a 30, 60° right triangular waveguide.

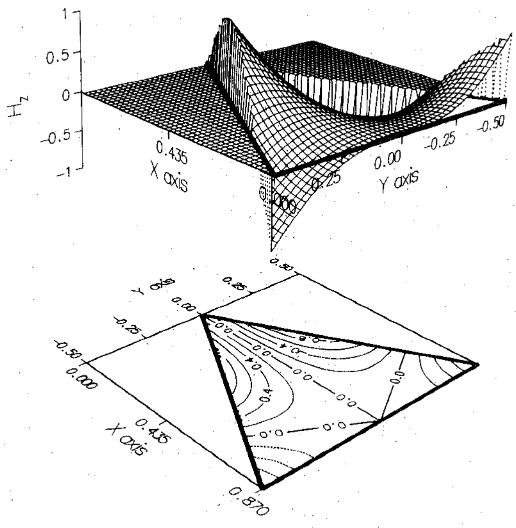

Fig. 5. $TE_{01}^{(2)}$ mode for an equilateral triangular waveguide (odd solution).

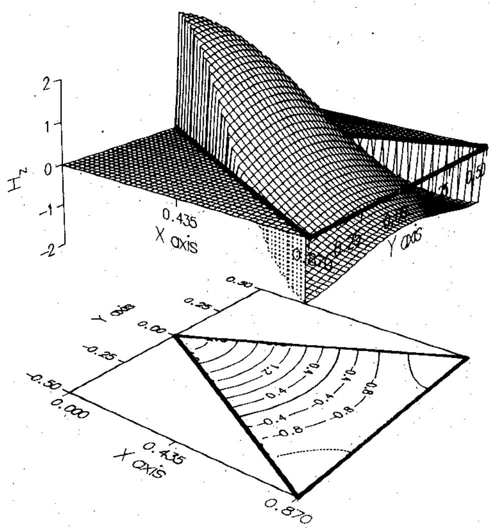

Fig. 4. $TE_{01}^{(1)}$ mode for an equilateral triangular waveguide (even solution).

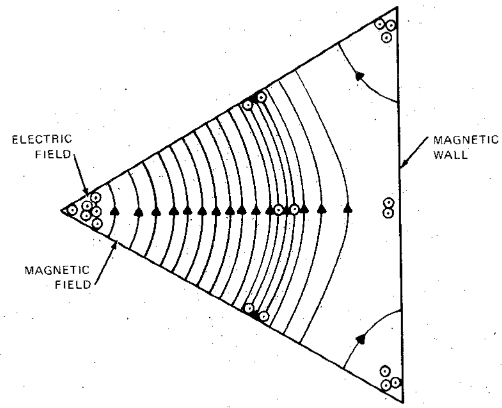

Fig. 6. $TM_{1,0,-1}$ dominant-mode field pattern in triangular resonator with magnetic walls [17].

waveguide cross sections are

$$30, 60° \text{ right:} \quad (k_c)^2_{mn} = \pi^2(m^2 + n^2/3)/a^2 \qquad (15)$$

$$\text{Equilateral:} \quad (k_c)^2_{mn} = (4\pi)^2(m^2 + n^2/3)/3a^2 \qquad (16)$$

$$30,30,120°: \quad (k_c)^2_{mn} = 4\pi^2(m^2 + n^2/3)/a^2. \qquad (17)$$

Values for the four lowest modes in each waveguide geometry are given in Table III.

III. Results

Contour and three-dimensional plots for the lowest order modes in the 30,60° right and equilateral triangular waveguides are shown in Figs. 3–5. The 30,60° right triangular TE_{11} mode is shown in Fig. 3. The TE_{01} mode does not exist due to the restrictions placed on the integers m and n by the boundary conditions, i.e., $(m+n)$ and $(m-n)$ even. Also, from Table III it is seen that the TE_{02} mode and the TE_{11} mode are the same. That modes with completely different indices have the same magnitude distribution and phase velocity is unique to triangular geometries and is a consequence of the nonorthogonal nature of the triangular solutions.

Figs. 4 and 5 represent the even and odd dominant modes, respectively, for the equilateral triangular waveguide. The even mode is an extremum, while the odd mode is zero along $y = 0$. These two completely different modes have the same phase velocity. Although this cross section has three-fold symmetry, the modes are even and odd with respect to only one of the surface perpendicular bisectors and are a mixture of odd and even modes about the remaining two.

Fig. 6 is a contour plot of the lowest order even mode for a triangular resonator with magnetic walls [17] obtained using Schelkunoff's lowest order TE solution for the equilateral triangular waveguide (by duality). The previously unrecognized existence of odd modes is discussed in the Appendix.

IV. Conclusions

A general approach for solving propagation problems in a certain class of waveguide cross sections based solely on Snell's law and superposition of plane waves has been presented. Exact eigenfunctions and eigenvalues have been determined for four waveguides with triangular cross sections. The transverse electric and magnetic mode solutions presented in Table I satisfy the standard Neumann and Dirichlet boundary conditions, respectively. These solutions are actually nonseparable solutions of the Helmholtz equation [18], [19]. For the special geometries considered, they reduce to finite sums of separable rectangular harmonics. For the equilateral triangular waveguide, it has been shown that the previous analytic solution has not provided a complete mode set and that the odd modes with respect to one of the symmetry axes are missing.

TABLE IV
$T(x, y)$ AND $H_z^{(1)}$: COMPARISON BETWEEN MODE INDICES

Case 1	Case 2	Case 3
If	If	If
$2k_1 = \frac{2\pi(m+n)}{a\sqrt{3}}$,	$2k_1 = \frac{2\pi m}{a\sqrt{3}}$,	$2k_1 = \frac{2\pi n}{a\sqrt{3}}$,
then	then	then
$2k_2 = \frac{2\pi(m-n)}{3a}$	$2k_2 = \frac{2\pi(m+2n)}{3a}$	$2k_2 = \frac{2\pi(2m+n)}{3a}$
and	and	and
$m' = \frac{m+n}{2}$ [a]	$m' = \frac{m}{2}$	$m' = \frac{n}{2}$
$n' = \frac{m-n}{2}$	$n' = n + \frac{m}{2}$	$n' = m + \frac{n}{2}$
$m = m'+n'$	$m = 2m'$	$m = n'-m'$
$n = m'-n'$	$n = n'-m'$	$n = 2m'$
$\frac{2\pi m}{a\sqrt{3}} = k_3$	$\frac{2\pi(m+n)}{a\sqrt{3}} = k_3$	$\frac{2\pi(m+n)}{a\sqrt{3}} = k_3$
$\frac{2\pi(m+2n)}{3a} = -k_4$	$\frac{2\pi(m-n)}{3a} = -k_4$	$\frac{2\pi(m-n)}{3a} = k_4$
$\frac{2\pi n}{a\sqrt{3}} = k_5$	$\frac{2\pi n}{a\sqrt{3}} = -k_5$	$\frac{2\pi m}{a\sqrt{3}} = -k_5$
$\frac{2\pi(2m+n)}{3a} = k_6$	$\frac{2\pi(2m+n)}{3a} = k_6$	$\frac{2\pi(m+2n)}{3a} = k_6$
m and n both odd or both even.	m even, n even or odd.	m even or odd, n even.

[a] Primed indices are ours; unprimed indices are Schelkunoff's.

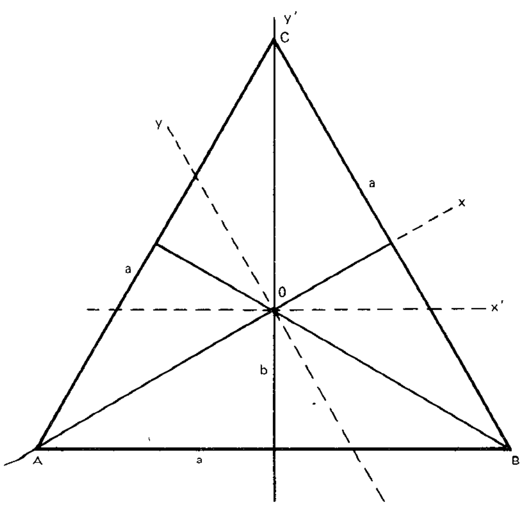

Fig. 7 Alternative coordinate systems for the equilateral triangular waveguide.

APPENDIX

Schelkunoff [6] finds both transverse electric (TE) and transverse magnetic (TM) mode solutions for the equilateral triangular waveguide. For TE waves, he gives a T function (equivalent to our H_z) of

$$T(x,y) = \cos\frac{2\pi l}{3b}\left(\frac{u}{2}+b\right)\cos\frac{\pi(m-n)(v-w)}{9b} + \cos\frac{2\pi m}{3b}\left(\frac{u}{2}+b\right)\cos\frac{\pi(n-l)(v-w)}{9b} + \cos\frac{2\pi n}{3b}\left(\frac{u}{2}+b\right)\cos\frac{\pi(l-m)(v-w)}{9b} \quad \text{(A1)}$$

where l, m, and n are integers obeying the relation

$$m+n+l=0, \qquad b=\frac{a}{2\sqrt{3}} \quad \text{(A2)}$$

and

$$u = x\cos\alpha + y\sin\alpha$$

$$v = x\cos\beta + y\sin\beta, \qquad \beta = \alpha + \frac{2\pi}{3}$$

$$w = x\cos\gamma + y\sin\gamma, \qquad \gamma = \beta + \frac{2\pi}{3}.$$

Fig. 7, after Schelkunoff, shows the equilateral triangle, its axes of symmetry and some possible coordinate systems. The angles α, β, and γ are the angles made by $A0$, $B0$, and $C0$, respectively, with the x-axis of a Cartesian coordinate system.

Choosing the $A0$ line as the x-axis, $\alpha = 0$, and with $l = -(m+n)$, (A1) becomes

$$T(x,y) = \cos\left[\left(\frac{2\pi x}{a\sqrt{3}} + \frac{2\pi}{3}\right)(m+n)\right]\cos\frac{2\pi(m-n)y}{3a} + \cos\left[m\left(\frac{2\pi x}{a\sqrt{3}} + \frac{2\pi}{3}\right)\right]\cos\frac{2\pi(m+2n)y}{3a} + \cos\left[n\left(\frac{2\pi x}{a\sqrt{3}} + \frac{2\pi}{3}\right)\right]\cos\frac{2\pi(2m+n)y}{3a}. \quad \text{(A3)}$$

This coordinate system is related to that used in this paper by a simple translation (see Fig. 2), which, when applied to solutions of this paper, gives

$$H_z^{(1)} = \cos 2k_1\left(x + \frac{a}{\sqrt{3}}\right)\cos 2k_2 y + \cos k_3\left(x + \frac{a}{\sqrt{3}}\right)\cos k_4 y + \cos k_5\left(x + \frac{a}{\sqrt{3}}\right)\cos k_6 y \quad \text{(A4a)}$$

$$H_z^{(2)} = \cos 2k_1\left(x + \frac{a}{\sqrt{3}}\right)\sin 2k_2 y - \cos k_3\left(x + \frac{a}{\sqrt{3}}\right)\sin k_4 y - \cos k_5\left(x + \frac{a}{\sqrt{3}}\right)\sin k_6 y \quad \text{(A4b)}$$

where $k_1 - k_6$ are given in Table I.

Comparing (A3) and (A4), it is seen that Schelkunoff has given only the even modes $H_z^{(1)}$. Shown in Table IV are possible relations between Schelkunoff's mode indices and those used in this paper.

Choosing the axes differently (e.g., x', y' in Fig. 7, where $\alpha = 30°$) such that x or y lies along a different symmetry axis ($A0$, $B0$, or $C0$), the modes will be expressed as a mixture of components, odd and even with respect to that axis. However, the intrinsic mode symmetry is with respect to the line $A0$ independent of the coordinate system chosen.

The equilateral triangular geometry has been proposed for waveguide Y circulators [20] and triangular resonators [17], [21]. In both [17] and [20], Schelkunoff's results in (A3) were used. It is suggested that the existence of the odd $H_z^{(2)}$ modes as well as the even $H_z^{(1)}$ modes is important in actual device applications.

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A GaAs MESFET Self-Bias Mode Oscillator

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***Abstract*— A self-bias mode oscillation in a GaAs MESFET, with the gate terminal kept open in a dc manner, has been analyzed by a large-signal MESFET circuit model. The circuit simulation demonstrates that the gate-source Schottky barrier becomes self-biased along with the microwave oscillation build-up and that a stable self-bias gate voltage is observed with a steady-state oscillation. A self-bias mode oscillator, operable with a single positive dc bias, is realized by using microwave integrated circuit technology.**

I. Introduction

GaAs metal-semiconductor field-effect transistor (MESFET) oscillator behavior has been investigated by using MESFET analytical models, including an intrinsic FET and on-chip and

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